Self-dual teleparallel formulation of general relativity and the positive energy theorem

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A self-dual and anti-self-dual decomposition of the teleparallel formulation of Einstein's general relativity is carried out and the self-dual Lagrangian of the teleparallel formulation of Einstein's general relativity, which is equivalent to the Ashtekar Lagrangian in vacuum, is obtained. Its Hamiltonian formulation and the constraint analysis are developed. Starting from Witten's equation the gauge condition of Nester and co-workers is derived directly and a new expression of the boundary term is obtained. Using this expression and Witten's identity the proof of the positive energy theorem by Nester and co-workers is extended to a case including momentum.

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I. INTRODUCTION

Recently, a specific teleparallel gravitational theory, the teleparallel formulation of Einstein's general relativity (TFGR), has attracted renewed attention [1–5] owing to its many salient features. First of all, teleparallel gravity can be regarded as a translational gauge theory [1,2,4,6,7], which makes it possible to unify gravity with other kinds of interactions in the gauge theory framework in which the elementary interactions are described by a connection defined on some principal fiber bundle. In this direction interesting developments have been achieved in the context of Ashtekar variables [8]. Mielke [7] used the teleparallel geometry of TFGR to give a transparent description of Ashtekar's new variables and a proof of the positivity of total energy for Einstein's theory in terms of the new variables. Moreover, it is proved that only for the above-mentioned choice can the Schwarzschild and the Reissner-Nordström black hole configurations be recovered [1] and the Dirac spinor field be consistently coupled to e^I_μ [2]. However, when dealing with supergravity and in the search for the construction of a unified model for the fundamental interactions, another usual route [9] is to consider Kaluza-Klein type models [10] of supergravity as candidates. In this approach the fields describing the fundamental interactions (including gravity) correspond to different pieces of the pseudo-Riemannian metric characterizing a higher dimensional spacetime. This approach is considered as quite different from the former and the relation between them has not been very clear. It is noteworthy that some teleparallel equivalents of the Kaluza-Klein theory and non-Abelian Kaluza-Klein theory are developed [11], which gives us new perspectives for the study of unified theories.

Another advantage of TFGR concerns energy momentum, its representation, positivity and localization [1,2,5]. Because of its simplicity and transparency TFGR seems to be much more appropriate than general relativity to deal with the problem of the gravitational energy momentum. It is proved that [1,2,5,12] in TFGR there exists a gravitational energy-momentum tensor which is covariant under general coordinate transformations and global Lorentz transformations.

Attempts at identifying an energy-momentum density for gravity in the context of general relativity led only to various energy-momentum complexes that are pseudotensors and then a new quasilocal approach that can be traced back to the early work of Penrose [13] was proposed and became widely accepted [5,14]. According to this approach a quasilocal energy momentum can be obtained from the Hamiltonian. Every energy-momentum pseudotensor is associated with a legitimated Hamiltonian boundary term. In terms of TFGR a geometrically natural proof of the positivity of the gravitational energy is obtained [5] by choosing a maximal surface and a vanishing shift.

Nester and co-workers have found a four-spinor formulation of TFGR [5]. This formulation has several virtues; in particular, it gives a four-covariant Hamiltonian and shows that the total four-momentum is future timelike and can be evaluated on a spacelike surface extending to future null infinity, thereby showing that the Bondi four-momentum is also future timelike. It is suggested to generalize this formulation to self-dual representations. The chiral Lagrangian formulation of general relativity employing two-component spinors has been introduced [15]. It is well known, as a self-dual formulation of general relativity, that Ashtekar's theory opens new avenues to quantum gravity and plays an important role in the development of modern gravitational theory. It is shown that a lot of gauge theories, gravity and supergravity theories have their self-dual partners [16–18]. A question naturally arises whether there is a self-dual teleparallel gravitational theory which is equivalent to Ashtekar's theory. If it exists, can it give us some new perspectives? A tensorial expression of the self- and anti-self-dual Lagrangian of TFGR, has been given by Mielke [7]. In this paper a self-dual TFGR, which starts from a two-spinor expression of the Lagrangian given in [7] and is equivalent to the Ashtekar theory [8], will be developed.

In the proof of the positive energy theorem [19] Witten proposes a spatial Dirac equation. In the last two decades people have been trying to understand the meaning of this equation and its solutions. In terms of an orthonormal frame Nester and co-workers gave another proof of the positive energy theorem [5] using teleparallel geometry under a special gauge in which the shift vanishes. Some authors found the relation between the orthonormal frame (triad) and the Witten equation [20]. In this paper gauge condition of Nester and co-workers will be derived from Witten's equation. Furthermore, a proof of the positive energy theorem different from Nester and co-workers by a nonvanishing shift and without maximal surface will be suggested.

In Sec. II the self-dual and anti-self-dual decomposition of the Lagrangian of TFGR is carried out and the self-dual teleparallel Lagrangian is given. In Sec. III the Hamiltonian formulation and the constraint algebra of the self-dual TFGR is built up. Its boundary term is just the self-dual part of the boundary term of Nester and co-workers. In Sec. IV, using Witten's equation the gauge condition of Nester and co-workers is derived and a new expression of the boundary term is obtained. Using this expression and the Witten identity a proof of the positive energy theorem is shown under a deferent gauge condition from the one of Nester and co-workers in Sec. V. Finally, Sec. VI is devoted to some conclusions.

II. SELF-DUAL AND ANTI-SELF-DUAL DECOMPOSITION OF THE LAGRANGIAN OF THE TELEPARALLEL FORMULATION OF GENERAL RELATIVITY

We start with a common relation between the tetrad $e^I{}_\mu$, the spin connection $\omega_\mu{}^I{}_J$, and the affine connection $\mathring{\Gamma}^\rho{}_{\nu\mu}$ [1,21]

$$\partial_\mu e^I_\nu + \omega_\mu{}^I{}_J e^J{}_\nu - \mathring{\Gamma}^\rho{}_{\mu\nu} e^I{}_\rho = 0, \tag{1}$$

where $I,J,\ldots=0,1,2,3$ are the internal indices and $\mu,\nu,\ldots=0,1,2,3$ are the spacetime indices. Defining the Weitzenbok connection [1,21]

$$\Gamma^\rho{}_{\mu\nu} = e_I{}^\rho \partial_\mu e^I{}_\nu, \tag{2}$$

then Eq. (1) leads to

$$\Gamma^\rho{}_{\mu\nu} = \left\{ {}^{\rho}_{\mu\nu} \right\} + \mathring{K}^\rho{}_{\mu\nu} - \omega_\mu{}^\rho{}_\nu, \tag{3}$$

where

$$\omega_\mu{}^\rho{}_\nu = \omega_\mu{}^I{}_J e_I{}^\rho e^J{}_\nu, \tag{4}$$

and $\{{}^{\rho}_{\mu\nu}\}$, $\mathring{K}^\rho{}_{\mu\nu}$ is the Christoffel connection and the affine contortion, respectively. By introducing the Weitzenbok torsion

$$T^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu} \tag{5}$$

and the Weitzenbok contortion [1,22]

$$K^\rho{}_{\mu\nu} = \frac{1}{2}(T^\rho{}_{\nu\mu} + T_\mu{}^\rho{}_\nu + T_\nu{}^\rho{}_\mu), \tag{6}$$

we can obtain from Eq. (3)

$$T^\rho{}_{\mu\nu} = \mathring{T}^\rho{}_{\mu\nu} - \omega_\mu{}^\rho{}_\nu + \omega_\nu{}^\rho{}_\mu, \tag{7}$$

where

$$\mathring{T}^\rho{}_{\mu\nu} = 2\mathring{\Gamma}^\rho{}_{[\mu\nu]} = \mathring{\Gamma}^\rho{}_{\mu\nu} - \mathring{\Gamma}^\rho{}_{\nu\mu} \tag{8}$$

is the affine torsion.

In this paper we concentrate on a specific teleparallel theory, the so-called teleparallel formulation of general relativity (TGFR) [3,22], which corresponds to a set of specific values of the coupling constants (Einstein choice) [7]. The Lagrangian of the gravitational field is

$$\mathcal{L}_G = -\frac{{}^{(4)}e}{8}(T_{\mu\nu\lambda}T^{\mu\nu\lambda} + 2T_{\mu\nu\lambda}T^{\lambda\nu\mu} - 4T^\mu{}_{\mu\lambda}T^\nu{}_\nu{}^\lambda), \tag{9}$$

which coincides, modulo a four-divergence, with the Einstein-Hilbert Lagrangian, where ${}^{(4)}e = \det(e^I{}_\mu)$. In TGFR the affine torsion vanishes

$$\mathring{T}^\rho{}_{\mu\nu} = 0, \tag{10}$$

and then Eq. (7) becomes

$$T^\rho{}_{\mu\nu} = \omega_\nu{}^\rho{}_\mu - \omega_\mu{}^\rho{}_\nu. \tag{11}$$

In the two-spinor formulation [23] the connection $\omega_{\mu\nu}{}^\rho = \omega_{AA'BB'}{}^{CC'}$ can be decomposed into two parts:

$$\begin{aligned}\omega_{AA'BB'}{}^{CC'} &= \omega_{AA'B}{}^C \epsilon^{C'}{}_{B'} + \overline{\omega}_{AA'B'}{}^{C'} \epsilon^C{}_B \\ &= \overset{+}{\omega}_{AA'BB'}{}^{CC'} + \overset{-}{\omega}_{AA'BB'}{}^{CC'},\end{aligned} \tag{12}$$

where

$$\overset{+}{\omega}_{AA'BB'}{}^{CC'} = \omega_{AA'B}{}^C \epsilon^{C'}{}_{B'}, \tag{13}$$

and

$$\overset{-}{\omega}_{AA'BB'}{}^{CC'} = \overline{\omega}_{AA'B'}{}^{C'} \epsilon^C{}_B \tag{14}$$

is the self-dual and the anti-self-dual part of the connection $\omega_{AA'BB'}{}^{CC'}$ $(A,B,\ldots=0,1;A',B',\ldots=0',1')$, respectively. Using these results we obtain

$$\begin{aligned}T_{(c)\mu\nu\lambda}T^{\mu\nu\lambda}_{(c)} &= T_{AA'BB'CC'}T^{AA'BB'CC'} \\ &= 4\omega_{AA'BC}\omega^{AA'BC} \\ &\quad + 4\overline{\omega}_{AA'B'C'}\overline{\omega}^{AA'B'C'} \\ &\quad - 2\omega_{AB'CB}\omega^{BB'AC} \\ &\quad - 2\overline{\omega}_{BA'C'B'}\overline{\omega}^{BB'A'C'} \\ &\quad + 4\omega_{AA'}{}^{AB}\overline{\omega}_{BB'}{}^{B'A'},\end{aligned}$$

$$T_{(c)\mu\nu\lambda}T_{(c)}^{\lambda\nu\mu}=-2\omega_{AA'BC}\omega^{AA'BC}-2\bar{\omega}_{AA'B'C'}\bar{\omega}^{AA'B'C'}+3\omega_{AB'CB}\omega^{BB'AC}+3\bar{\omega}_{BA'C'B'}\bar{\omega}^{BB'A'C'}-6\omega_{AA'}{}^{AB}\bar{\omega}_{BB'}{}^{B'A'},$$

and

$$T^{\mu}_{(c)\mu\lambda}T^{\nu}_{(c)\nu}{}^{\lambda}=\omega_{AA'}{}^{AC}\omega^{AB'}{}_{AC}+\bar{\omega}_{AA'}{}^{A'C'}\bar{\omega}^{AB'}{}_{B'C'}-2\omega_{AA'}{}^{AB}\bar{\omega}_{BB'}{}^{B'A'}.$$

The Lagrangian $\mathcal{L}_G$ takes the form

$$\mathcal{L}_G=-\frac{{}^{(4)}\sigma}{4}(4\omega_{AB'CB}\omega^{BB'AC}+4\bar{\omega}_{BA'C'B'}\bar{\omega}^{BB'A'C'}-4\omega_{AA'}{}^{AC}\omega^{AB'}{}_{AC}-4\bar{\omega}_{AA'}{}^{A'C'}\bar{\omega}^{AB'}{}_{B'C'})$$
$$={}^{(4)}\sigma(\omega_{AA'}{}^{AC}\omega^{BA'}{}_{BC}+\bar{\omega}_{AA'}{}^{A'C'}\bar{\omega}^{AB'}{}_{B'C'}-\omega_{AB'CB}\omega^{BB'AC}-\bar{\omega}_{BA'C'B'}\bar{\omega}^{BB'A'C'}),\qquad(15)$$

and splits into two parts:

$$\mathcal{L}_G=\mathcal{L}_G^{+}+\mathcal{L}_G^{-},\qquad(16)$$

where

$$\mathcal{L}_G^{+}={}^{(4)}\sigma(\omega_{AA'}{}^{AC}\omega^{BA'}{}_{BC}-\omega_{AB'CB}\omega^{BB'AC})\qquad(17)$$

and

$$\mathcal{L}_G^{-}={}^{(4)}\sigma(\bar{\omega}_{AA'}{}^{A'C'}\bar{\omega}^{AB'}{}_{B'C}-\bar{\omega}_{BA'C'B'}\bar{\omega}^{BB'A'C'})\qquad(18)$$

is the self-dual part and the anti-self-dual part of $\mathcal{L}_G$ with the determinant ${}^{(4)}\sigma$ of the inverse SL(2,C) soldering form $\sigma_{\mu}{}^{AA'}$ on the spacetime manifold M. $\mathcal{L}_G^{+}$ and $\mathcal{L}_G^{-}$ are the two-spinor expressions of the chiral Lagrangian $V_{\parallel}^{(+)}$ and $V_{\parallel}^{(-)}$ given by Mieke [7], respectively.

Since $\mathcal{L}_G$ consists of two invariant parts, depending on $\omega_{AA'B}{}^{C}$ and $\bar{\omega}_{AA'B'}{}^{C'}$, respectively, we can choose the self-dual part $\mathcal{L}_G^{+}$ as the Lagrangian which is the equivalent of the Ashtekar Lagrangian [8], modulo a four-divergence [7].

III. THE HAMILTONIAN FORMULATION OF THE SELF-DUAL TELEPARALLEL FORMULATION OF GENERAL RELATIVITY

In order to obtain the Hamiltonian of the theory a foliation in the spacetime manifold M should be introduced. Assuming that $M=\Sigma\times R$ for some space-like manifold Σ, we can choose a time function t with nowhere vanishing gradient $(dt)_{\mu}$ such that each $t=$const surface Σ_t is diffeomorphic to Σ. We introduce a time flow vector t^{μ} satisfying $t^{\mu}(dt)_{\mu}=1$, and we can decompose it perpendicular and parallel to Σ_t: $t^{\mu}=Nn^{\mu}+N^{\mu}$, where n^{μ} is the time-like normal at each point of Σ_t and N, N^{μ} are the lapse function and the shift vector, respectively. The spacetime metric $g_{\mu\nu}$ introduces a spatial metric $q_{\mu\nu}$ on each Σ_t by the formula

$$q_{\mu\nu}=g_{\mu\nu}+n_{\mu}n_{\nu}.\qquad(19)$$

In the two-spinor formalism the unit normal vector $n^{\mu}=n^{AA'}$ defines an isomorphism from the space of primed spinors to the space of unprimed spinors [8,24],

$$\xi^{A}=\sqrt{2}n^{AA'}\xi_{A'},$$
$$\omega_{ABCD}=\sqrt{2}n_{A}{}^{A'}\omega_{AA'CD},\qquad(20)$$

$$n^{AB}=\sqrt{2}n^{BA'}n^{A}{}_{A'}=\frac{1}{\sqrt{2}}\epsilon^{AB}.\qquad(21)$$

In this formalism Eq. (19) reads

$$g^{ABCD}=q^{ABCD}+n^{AB}n^{CD},\qquad(22)$$

or

$$\epsilon^{AC}\epsilon^{BD}=-\epsilon^{A(C}\epsilon^{D)B}+\frac{1}{2}\epsilon^{AB}\epsilon^{CD}.\qquad(23)$$

Using these results and decomposing ω_{ABCD} into its symmetry part $\omega_{(AB)CD}$ and skew-symmetry part $\omega_{[AB]CD}$,

$$\omega_{ABCD}=\omega_{(AB)CD}+\omega_{[AB]CD},\qquad(24)$$

the Lagrangian $\mathcal{L}_G^{+}$ can be written as

$$\mathcal{L}_G^{+}={}^{(4)}\sigma[\omega_{(AB)}{}^{AC}\omega^{DB}{}_{DC}-\omega_{(AB)CD}\omega^{CBAD}-\sqrt{2}\omega_{\perp CD}\omega^{(CE)}{}_{E}{}^{D}],$$

where

$$\omega_{\perp CD}=n^{AB}\omega_{ABCD}.\qquad(25)$$

From Eq. (1) one gets

$$\omega_{\mu}{}^{I}{}_{\nu}=\omega_{\mu}{}^{I}{}_{J}e^{J}{}_{\nu}=-\partial_{\mu}e^{I}_{\nu}+\Gamma^{\rho}{}_{\mu\nu}e^{I}{}_{\rho}:=-\mathring{\nabla}_{\mu}e^{I}{}_{\nu},\qquad(26)$$

where $\mathring{\nabla}_{\mu}$ is the affine covariant derivative which is just the Christoffel covariant derivative [1] since we have assumed the vanishing affine torsion. In the two-spinor formalism (26) reads

$$\omega_{CDAB}=-\zeta_{aA}\mathring{\nabla}_{CD}\zeta^{a}{}_{B}.$$

Using the relation

$$n^{AB}=\frac{1}{N}(t^{AB}-N^{AB}),$$

one gets

$$\omega_{\perp CD} = -\frac{1}{N}\zeta_C{}^b\,\dot{\zeta}_{bD} - \frac{1}{N}N^{AB}\omega_{ABCD}, \tag{27}$$

where

$$\dot{\zeta}_{bD} = t^{AB}\mathring{\nabla}_{AB}\zeta_{bD}. \tag{28}$$

The Lagrangian $\mathcal{L}_G^+$ becomes

$$\begin{aligned}\mathcal{L}_G^+ = {}^{(4)}\sigma[\omega_{(AB)}{}^{AC}\omega^{(DB)}{}_{DC} - \omega_{(AB)CD}\omega^{(CB)AD}] \\ - {}^{(4)}\sigma[\omega_{(AB)}{}^{AC}\zeta_D{}^a\mathring{\nabla}^{[DB]}\zeta_{aC} - \omega_{(AB)CD}\zeta^{Aa}\mathring{\nabla}^{[CB]}\zeta_a{}^D] \\ + {}^{(4)}\sigma\frac{\sqrt{2}}{N}(\zeta_C{}^b\,\dot{\zeta}_{bD} + N^{AB}\omega_{ABCD})\omega^{(CE)}{}_E{}^D.\end{aligned}$$

The second term can be rewritten

$$\begin{aligned}\omega_{(AB)}{}^{AC}\zeta_D{}^a\mathring{\nabla}^{[DB]}\zeta_{aC} - \omega_{(AB)CD}\zeta^{Aa}\mathring{\nabla}^{[CB]}\zeta_a{}^D \\ = -\frac{\sqrt{2}}{N}\omega_{(AB)}{}^{AC}(\zeta^{Ba}\dot{\zeta}_{aC} + N^{EF}\omega_{EF}{}^B{}_C)\end{aligned}$$

and then one obtains

$$\begin{aligned}\mathcal{L}_G^+ = N\sigma[\omega_{(AB)}{}^{AC}\omega^{(DB)}{}_{DC} - \omega_{(AB)CD}\omega^{(CB)AD}] \\ + 2\sqrt{2}\sigma(\zeta_C{}^b\,\dot{\zeta}_{bD} + N^{AB}\omega_{ABCD})\omega^{(CE)}{}_E{}^D,\end{aligned} \tag{29}$$

where

$$\sigma = \frac{{}^{(4)}\sigma}{N} = \det\sigma_\mu{}^{AB} = \frac{1}{N}\sqrt{-g}. \tag{30}$$

The canonical momentum conjugate to $\dot{\zeta}_{bD}$ is

$$\begin{aligned}\tilde{p}^{bD} = \frac{\partial\mathcal{L}_G^+}{\partial\dot{\zeta}_{bD}} \\ = 2\sigma\sqrt{2}\zeta_C{}^b\omega^{(CE)}{}_E{}^D.\end{aligned} \tag{31}$$

Here $\omega_{(AB)}{}^{CD}$ is just Ashtekar's variable which appears in the canonical momentum conjugate $\tilde{p}^{bD}$ to ζ_{bD} and is related to $\tilde{p}^{bD}$ by

$$\omega^{(AC)}{}_C{}^B = -\frac{1}{2\sigma\sqrt{2}}\zeta_a{}^A\tilde{p}^{aB}. \tag{32}$$

This result is similar to the result given by Mielke [7] in which Ashtekar's variable is identified with the momentum canonically conjugate to the ‘‘triad densities.'' The gravitational Hamiltonian can be computed

$$\begin{aligned}\mathcal{H}_G = \tilde{p}^{bD}\,\dot{\zeta}_{bD} - \mathcal{L}_G^+ \\ = \sigma[N\mathcal{H}_\perp + N^{AB}\mathcal{H}_{AB} + \mathring{\nabla}_{(AB)}B^{AB}],\end{aligned}$$

where

$$\begin{aligned}\mathcal{H}_\perp = \omega_{(AB)CD}\omega^{(AB)CD} - \frac{1}{\sqrt{2}}\omega_{(AB)}{}^{AC}\zeta_b{}^B\tilde{p}^b{}_C \\ + \frac{1}{2\sqrt{2}}\mathring{\nabla}_{(AB)}(\zeta_b{}^B\tilde{p}^{bA}) + \frac{1}{2\sqrt{2}}\zeta_b{}^B\tilde{p}^{bA}\mathring{\nabla}_{(AB)}\ln N \\ - \frac{\sqrt{2}}{N}\omega^{(AB)}{}_{EC}\mathring{\nabla}_{(AB)}N^{CE},\end{aligned} \tag{33}$$

$$\begin{aligned}\mathcal{H}_{AB} = \sqrt{2}\sigma[-\mathring{\nabla}_{(CD)}\omega^{(CD)}{}_{AB} + 2\omega^{(CD)}{}_{BE}\omega_{(CD)A}{}^E] \\ - \zeta_b{}^C\tilde{p}^{bD}\omega_{(AB)CD},\end{aligned} \tag{34}$$

and

$$\begin{aligned}\tilde{B}^{(AB)} = -\frac{1}{2}(N\sigma\omega^{(CB)A}{}_C + N\sigma\omega^{(CA)B}{}_C) \\ - \sqrt{2}\sigma N^{CD}\omega^{(AB)}{}_{DC}.\end{aligned} \tag{35}$$

Here $\tilde{B}^{(AB)}$ is just the self-dual part of the boundary term $\tilde{B}^\mu$ given by Nester and co-workers [5].

By following the Dirac constraint analysis we find that the theory has the same constraint structure. There are only two constraints, the scalar constraint

$$\mathcal{H}_\perp = 0,$$

and the vector constraint

$$\mathcal{H}_{AB} = 0.$$

The phase space $(\Gamma_{TG}, \Omega_{TG})$ of the teleparallel gravity is coordinated by the pair $(\zeta_{bD}, \tilde{p}^{bD})$ and has symplectic structure

$$\Omega_{TG} = \int_\Sigma d\tilde{p}^{bD} \wedge d\zeta_{bD}. \tag{36}$$

By constructing the constraint functions by smearing $\mathcal{H}_\perp$ and $\mathcal{H}_i$ with test fields N and N^i on Σ following the approach of Ashtekar [8],

$$C(N) = \int_\Sigma N\mathcal{H}_\perp,$$

$$C(\vec{N}) = \int_\Sigma N^{AB}\mathcal{H}_{AB},$$

we find that in the case $\partial_i N^i = 0$, the constraint algebra is given by

$$\{C(N), C(M)\} = C(\mathcal{L}_{\vec{t}}M) - C(\mathcal{L}_{\vec{t}}M), \tag{37}$$

$$\{C(N), C(M) = C(\mathcal{L}_{\vec{N}}M), \tag{38}$$

and

$$\{C(\vec{N}), C(\vec{M}) = C(\mathcal{L}_{\vec{N}}\vec{M}) = C([\vec{N}, \vec{M}]). \tag{39}$$

These equations indicate that the constraint algebra is closed and the constraints $C(N)$ and $C(N)$ are first class, which is very similar to the case in general relativity. The first class constraints $C(N)$ and $C(\vec{N})$ generate the corresponding gauge transformations, the spacetime translations. We have shown that the constraint algebra of the teleparallel gravity has the same structure as that of general relativity.

IV. WITTEN-NESTER GAUGE CONDITIONS

Introducing the Lorentz covariant derivative of $e^I{}_\nu$ by

$$\nabla_\mu e^I{}_\nu = \partial_\mu e^I_\nu + \omega^I_{\mu J} e^J_\nu,$$

then we have

$$\omega_\mu{}^\rho{}_\nu = e_I{}^\rho e^J{}_\nu \omega_\mu{}^I{}_J = e_I{}^\rho (\nabla_\mu e^I{}_\nu - \partial_\mu e^I_\nu).$$

Using the dyad

$$\zeta_{0A} = o_A, \quad \zeta_{1A} = \iota_A, \quad \zeta^{A0} = -\iota^A, \quad \zeta^{A1} = o^A, \tag{40}$$

and supposing

$$o^A = \frac{1}{\chi}\lambda^A, \quad \iota^A = \frac{1}{\chi}\lambda^{\dagger A}, \tag{41}$$

one obtains

$$\begin{aligned}\omega_{CDAB} &= \zeta_{aA}(\nabla_{CD}\zeta^a{}_B - \partial_{CD}\zeta^a{}_B)\\ &= \frac{1}{\chi^2}(\lambda^\dagger_A \nabla_{CD}\lambda_B - \lambda_A \nabla_{CD}\lambda^\dagger_B - \lambda^\dagger_A \partial_{CD}\lambda_B\\ &\quad + \lambda_A \partial_{CD}\lambda^\dagger_B),\end{aligned} \tag{42}$$

and then

$$\begin{aligned}\omega^{(CB)A}{}_C + \omega^{(CA)B}{}_C = \frac{1}{\chi^2}(&\lambda^{\dagger A}\nabla^{(CB)}\lambda_C - \lambda^A \nabla^{(CB)}\lambda^\dagger_C\\ &+ \lambda^{\dagger B}\nabla^{(CA)}\lambda_C - \lambda^B \nabla^{(CA)}\lambda^\dagger_C\\ &- \lambda^{\dagger A}\partial^{(CB)}\lambda_C + \lambda^A \partial^{(CB)}\lambda^\dagger_C\\ &- \lambda^{\dagger B}\partial^{(CA)}\lambda_C + \lambda^B \partial^{(CA)}\lambda^\dagger_C).\end{aligned}$$

Suppose the spinors λ^A and its conjugate $\overline{\lambda^{A'}}$ are the solutions of the Witten equation

$$\nabla_{(AB)}\lambda^A = 0, \tag{43}$$

and

$$\nabla_{(A'B')}\overline{\lambda^{A'}} = 0. \tag{44}$$

The latter leads to

$$\nabla_{(AB)}\lambda^{\dagger A} = \frac{1}{\sqrt{2}} K \lambda^\dagger_B. \tag{45}$$

Using Eqs. (40) and (42) one can compute

$$\begin{aligned}\omega^{(CB)A}{}_C + \omega^{(CA)B}{}_C = &-\frac{1}{\sqrt{2}\chi^2} K(\lambda^B\lambda^{\dagger A} + \lambda^A\lambda^{\dagger B})\\ &- \frac{1}{\chi^2}(\lambda^{\dagger A}\partial^{(CB)}\lambda_C - \lambda^A\partial^{(CB)}\lambda^\dagger_C\\ &+ \lambda^{\dagger B}\partial^{(CA)}\lambda_C - \lambda^B\partial^{(CA)}\lambda^\dagger_C).\end{aligned} \tag{46}$$

Introducing the triad on the spacelike hypersurface Σ:

$$\begin{aligned}e_1{}^{AB} &= \frac{1}{\sqrt{2}}(m^a - \overline{m}^a) = \frac{1}{\sqrt{2}\chi^2}(\lambda^A\lambda^B + \lambda^{\dagger A}\lambda^{\dagger B}),\\ e_2{}^{AB} &= \frac{-i}{\sqrt{2}}(m^a + \overline{m}^a) = \frac{-i}{\sqrt{2}\chi^2}(\lambda^A\lambda^B - \lambda^{\dagger A}\lambda^{\dagger B}),\\ e_3{}^{AB} &= \frac{1}{\sqrt{2}}(l^a - n^a) = \frac{1}{\sqrt{2}\chi^2}(\lambda^A\lambda^{\dagger B} + \lambda^{\dagger A}\lambda^B),\end{aligned} \tag{47}$$

one can compute

$$\begin{aligned}\partial^{(AB)}e_{1AB} = &-2\chi^{-1}\partial^{(AB)}\chi e_{1AB} + \frac{\sqrt{2}}{\chi^2}(\lambda_A\partial^{(AB)}\lambda_B\\ &+ \lambda^\dagger_A\partial^{(AB)}\lambda^\dagger_B),\end{aligned}$$

$$\begin{aligned}(\omega^{(CB)A}{}_C + \omega^{(CA)B}{}_C)e_{1AB} &= -\frac{\sqrt{2}}{\chi^2}(\lambda_A\partial^{(AB)}\lambda_B + \lambda^\dagger_A\partial^{(AB)}\lambda^\dagger_B)\\ &= -2\chi^{-1}\partial^{(AB)}\chi e_{1AB} - \partial^{(AB)}e_{1AB}\\ &= -\partial_1 \ln\chi^2 - \partial^{(AB)}e_{1AB}.\end{aligned}$$

By the same way one gets

$$(\omega^{(CB)A}{}_C + \omega^{(CA)B}{}_C)e_{2AB} = -\partial_2 \ln\chi^2 - \partial^{(AB)}e_{2AB},$$

$$(\omega^{(CB)A}{}_C + \omega^{(CA)B}{}_C)e_{3AB} = -\partial_3 \ln\chi^2 - \partial^{(AB)}e_{3AB} + K.$$

Then we have

$$\begin{aligned}q_I &= \omega^{+\,j}_{jI} = (\omega^{(CB)A}{}_C + \omega^{(CA)B}{}_C)e_{IAB}\\ &= -\partial_I \ln\chi^2 - \partial^{(AB)}e_{IAB} + \delta_{I3}K \quad (I=1,2,3).\end{aligned} \tag{48}$$

This is just the gauge condition of Nester and co-workers with a correct term $\partial^{(AB)}e_{IAB}$.

V. WITTEN IDENTITY AND POSITIVITY OF GRAVITATIONAL ENERGY

Using Eq. (42) one can compute

$$-N^j\omega^{+i}{}_{\perp j}=-\sqrt{2}N^{CD}\omega^{(AB)}{}_{DC}$$
$$=-\frac{\sqrt{2}}{\chi^2}N^{CD}[\lambda_C\nabla^{(AB)}\lambda^\dagger_D-\lambda^\dagger_C\nabla\partial^{(AB)}\lambda_D]$$
$$+\frac{\sqrt{2}}{\chi^2}N^{CD}[\lambda_C\partial^{(AB)}\lambda^\dagger_D-\lambda^\dagger_C\partial^{(AB)}\lambda_D].$$

Supposing

$$N^{CD}=-\frac{1}{\sqrt{2}}\lambda^{(\dagger C}\lambda^{D)}, \tag{49}$$

we have

$$-N^j\omega^{+i}{}_{\perp j}=\sqrt{2}N^{CD}n_C{}^E\omega^{(AB)}{}_{DE}$$
$$=\lambda^{\dagger C}\nabla^{(AB)}\lambda_C-\lambda^{\dagger C}\partial^{(AB)}\lambda_C.$$

If we choose

$$N=\lambda_A\lambda^{\dagger A}=\chi^2, \tag{50}$$

the integral of the boundary term (35) reads

$$\oint_S\tilde{B}^{(AB)}dS_{AB}=-\frac{1}{2}\oint_S\sigma(N\omega^{(CB)A}{}_C+N\omega^{(CA)B}{}_C)dS_{AB}$$
$$+2\oint_S\sigma N^{CD}n_C{}^E\omega^{(AB)}{}_{DE}dS_{AB}$$
$$=\oint_S\sigma\chi^2(\partial_I\ln\chi^2+\partial^{(AB)}e_{IAB})dS^I$$
$$+\oint_S\sigma K dS^3\oint_S\sigma\{\lambda^{\dagger C}\nabla^{(AB)}\lambda_C$$
$$-\lambda^{\dagger C}\partial^{(AB)}\lambda_C\}dS_{AB}. \tag{51}$$

Using the Witten identity [19]

$$\oint_S\sigma\lambda^{\dagger A}\nabla_i\lambda_A dS^i=2\int_\Sigma\sigma(\nabla^{(BC)}\lambda^A)^\dagger(\nabla_{(BC)}\lambda_A)dV$$
$$+4\pi G\int_\Sigma\sigma\lambda^{\dagger A}(T_{00}\lambda_A+\sqrt{2}T_{0AB}\lambda^B)dV, \tag{52}$$

one finds

$$\oint_S\tilde{B}^{(AB)}dS_{AB}=\frac{1}{2\sqrt{2}}\oint_S\sigma K(\lambda^B\lambda^{\dagger A}+\lambda^A\lambda^{\dagger B})dS_{AB}$$
$$+\frac{1}{2}\oint_S\sigma(\lambda^{\dagger A}\partial^{(CB)}\lambda_C-\lambda^A\partial^{(CB)}\lambda^\dagger_C$$
$$+\lambda^{\dagger B}\partial^{(CA)}\lambda_C-\lambda^B\partial^{(CA)}\lambda^\dagger_C-\lambda^{\dagger C}\partial^{(AB)}\lambda_C)$$
$$\times dS_{AB}2\int_\Sigma\sigma(\nabla^{(BC)}\lambda^A)^\dagger(\nabla_{(BC)}\lambda_A)dV$$
$$+4\pi G\int_\Sigma\sigma\lambda^{\dagger A}(T_{00}\lambda_A+\sqrt{2}T_{0AB}\lambda^B)dV, \tag{53}$$

which leads the positivity of the gravitational energy in the asymptotically flat boundary condition in space infinity. It is to be noted that we do not use the gauge $N^{AB}=0$ of Nester and workers. Instead we suppose the equation (49), which means that the proof of Nester and co-workers is extended to the case including momentum. The gauge condition of Nester and co-workers plays a role only in the lapse part of the boundary term, while the Witten equation plays roles not only in the lapse part but also in the shift part of the boundary term in the proof of the positive energy theorem.

VI. CONCLUSIONS

Expressing the chiral Lagrangian given by Mielke in terms of two-spinors, a self-dual teleparallel formulation of general relativity is developed. Its Lagrangian is equivalent to the Ashtekar Lagrangian. The basic dynamic variables are the dyad spinors ζ_{aA}. The Ashtekar connection appears in the canonical momentum $\tilde{p}^{aA}$ conjugate to ζ_{aA}. In the Hamiltonian formulation of this theory the gauge condition of Nester and co-workers can be derived from the Witten equation directly and a new expression for the boundary term, which is the self-dual part of the boundary term of Nester and co-workers, is obtained. Using this expression the proof of the positive energy theorem by Nester and co-workers can be extended to a case that includes momentum.

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